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AN EQUIVALENT MATHEMATICAL PROGRAM FOR MONOTONE VARIATIONAL INEQUALITY PROBLEMS SUBJECT TO LINEAR EQUALITY CONSTRAINTS

by

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Cette étude a été publiée grâce à une subvention du fonds F.C.A.R. pour l'aide et le soutien à la recherche.

Université de Montréal

Centre de recherche sur les transports - Publication #559
Département d'informatique et de
recherche opérationnelle - Publication #644

January 1988

Dépôt légal - 1er trimestre 1988

ABSTRACT

In this paper, we consider monotone variational inequality problems subject to linear equality constraints: VI(g,Ψ): find $x^* \in \Psi \mid g(x^*)^T(x-x^*) \geq 0$ for all $x \in \Psi = \{x \in \mathbb{R}^n \mid Ax=b\}$.

Usually, a monotone variational inequality problem cannot be formulated as an equivalent one level mathematical program unless the Jacobian ∇g is symmetric. In this paper, we show that the problem VI(g,Ψ) has an equivalent one level mathematical program P_Γ (even if the Jacobian ∇g is assymmetric): P_Γ: min $\Gamma(x)$ s.t. $x \in \Psi$ where $\Gamma^2(x)=g(x)^T Gg(x)$, and $G=I-A^T(AA^T)^{-1}A$ is the projection matrix on Ψ.

On the other hand, we show that the Newton direction for VI(g,Ψ) is a feasible descent direction for P_Γ, so this direction, at each iteration, may be used in combination with a step to reduce the objective value $\Gamma(x)$.

Key words: variational inequality; mathematical programming; Newton.

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RÉSUMÉ

Dans cet article, nous considérons les problèmes d'inégalité variationnelle monotone sujet à des contraintes d'égalité linéaires : VI(g,Ψ): trouver $x^* \in \Psi \mid g(x^*)^T(x-x^*) \geq 0$ pour tout $x \in \Psi = \{x \in \mathbb{R}^n \mid Ax=b\}$.

Habituellement, les problèmes d'inégalité variationnelle ne peuvent pas être exprimés comme des programmes mathématiques à un niveau, sauf si le Jacobien ∇g est symétrique. Dans cet article, nous montrons que le problème VI(g,Ψ) possède un programme mathématique à un niveau P_Γ équivalent (même si le Jacobien ∇g est non symétrique): P : min $\Gamma(x)$ s.à. x $\Gamma^2(x)=g(x)^T Gg(x)$, et $G=I-A^T(AA^T)^{-1}A$ étant la matrice de projection sur Ψ.

D'un autre côté, nous prouvons que la direction de Newton, pour la résolution de VI(g,Ψ), est une direction de descente réalisable pour P_Γ; ainsi, cette direction peut être utilisée en combinaison avec un pas de façon à réduire, à chaque itération, la valeur de l'objectif $\Gamma(x)$.

Mots-clés : inégalité variationnelle; programmation mathématique; Newton.

INTRODUCTION

In this paper, we consider monotone variational inequality VI(g,Ψ):

$$VI(g,\Psi): \text{ Find } x^*\in\Psi \mid g(x^*)^T(x-x^*) \geq 0 \quad \text{for all } x\in\Psi \tag{1}$$

$$\text{where } \Psi = \{x\in\mathbb{R}^n \mid Ax=b\} \tag{2}$$

The matrix A:mxn is supposed to be of full rank $m<n$, and b is a vector of dimension mx1, so Ψ is a (convex) polytope of $\mathbb{R}^n$.

The mapping $g:\mathbb{R}^n \to \mathbb{R}^n$ is supposed to be strictly monotone increasing and differentiable on Ψ.

The purpose of this paper is to prove that VI(g,Ψ) is equivalent to a mathematical program, even if the Jacobian ∇g is not symmetric. We recall that, according to the symmetry principle (Ortega and Rheinboldt, 1970), there exists no function $f:\mathbb{R}^n \to \mathbb{R}$ such that $\nabla f=g$ when ∇g is asymmetric, so VI(g,Ψ) cannot be written min f(x) s.t. $x\in\Psi$ as in the symmetric case (where $\nabla f=g$). This paper is not in contradiction with the symmetry principle because the mathematical program min $\Gamma(x)$ s.t. $x\in\Psi$, that we show to be equivalent to VI(g,Ψ), does <u>not</u> verify $\nabla\Gamma=g$. What we show is that these two problems have the same solution (that is in this paper supposed to exist).

Any of the efficient methods of mathematical programming may be used to solve the equivalent mathematical program, leading to the solution of the variational inequality VI(g,Ψ). We consider, in particular, the Newton method which is a contraction approach for solving VI(g,Ψ), and show that it is a descent method for the equivalent mathematical program. The usual Newton direction may, then, be used with a step and to reduce the value of the objective at each iteration.

The developments are given in the next sections.

Equivalence between VI(g,Ψ) and P_Γ

In this section, we consider the variational inequality (1)-(2), and the mathematical program P_Γ:

$$(P_\Gamma): \begin{cases} \min \Gamma(x) & (3) \\ \text{s.t. } x \in \Psi & (4) \end{cases}$$

where $\Gamma(x)$ is a gap function defined as follows:

$$\Gamma: \Psi \to \mathbb{R}$$

$$x \to \Gamma(x) = \begin{cases} \max\limits_{y} g(x)^T(y-\tilde{x}) & (5) \\ A(y-\tilde{x}) = 0 & (6) \\ (y-\tilde{x})^2 = 1 & (7) \end{cases}$$

where $\tilde{x} \in \Psi$ is choosen arbitrarily. Thus, if we adopt the notation: $d=y-\tilde{x}$, (5)-(7) may be rewritten:

$$\Gamma(x) = \begin{cases} \max\limits_{d} g(x)^T d & (9) \\ \text{s.t. } Ad = 0 & (10) \\ d^T d = 1 & (11) \end{cases}$$

$$\text{As in (2): } \Psi = \{x \in \mathbb{R}^n \mid Ax=b\}, \quad \Psi_d = \{d \in \mathbb{R}^n \mid Ad=0\}, \tag{12}$$

and it is clear that Ψ and Ψ_d are parallel.

In the following, we prove that VI(g,Ψ) is equivalent to P_Γ. Recall that there exists a solution to VI(g,Ψ).

Lemma 1: (i) $\Gamma(x) \geq 0$ for all $x \in \Psi$

(ii) $\Gamma(x^*) = 0$ where x^* is the optimal solution to VI(g,Ψ).

Proof : (i) By contradiction, suppose that $\Gamma(x) < 0$ for some $x \in \Psi$, and let $\tilde{d}$ be the optimal solution to (9)-(11). Then $g(x)^T \tilde{d} < 0$ and $A\tilde{d} = 0$, $\tilde{d}^T\tilde{d} = 1$. Thus $A(-\tilde{d}) = -A\tilde{d} = 0$ and $(-\tilde{d})^T(-\tilde{d}) = \tilde{d}^T\tilde{d} = 1$, so $-\tilde{d}$ is feasible. But $g(x)^T(-\tilde{d}) = -g(x)^T \tilde{d} > 0 > g(x)^T \tilde{d} = \Gamma(x)$ which is in contradiction with $\tilde{d}$ optimal. Then $\Gamma(x)$ is not negative; $\Gamma(x) \geq 0$.

(ii) Here we consider the optimal solution, $x^* \in \Psi$, to VI(g,Ψ). By (i) $\Gamma(x^*) \geq 0$. Again by contradiction, suppose that there exists $x^0 \in \Psi$ such that $g(x^*)^T(x^0-x^*) > 0$. Let $x' = 2x^*-x^0$. Then $Ax' = 2Ax^* - 2Ax^0 = 2b-b = b$ (because x^* and x^0 are in Ψ) is in Ψ, $x' \in \Psi$. And $g(x^*)^T(x'-x^*) = g(x^*)^T(2x^*-x^0-x^*) = g(x^*)^T(x^*-x^0) = -g(x^*)^T(x^0-x^*) < 0$ contradicting x^* optimal solution of VI(g,Ψ). Then such an $x^0 \in \Psi$ does not exist, and consequently, for all $x \in \Psi$, $g(x^*)^T(x-x^*) = 0$. (13)

Using the notation (8) with $\tilde{x}=x^*$ $d=x-x^*$ and $\Psi_d = \{d \mid Ad=0\}$, (13) becomes: for all $d \in \Psi_d$ $g(x^*)^T d = 0$ (14)

Recall that $\Gamma(x^*) = \begin{cases} \max & g(x^*)^T d \\ \text{s.t.} & d \in \Psi_d \\ & d^T d = 1 \end{cases}$

By using (14): $g(x^*)^T d = 0$ for any $d \in \Psi_d$, so in particular, $g(x^*)^T d = 0$ for any $d \in \Psi_d \cap \{d \mid d^T d = 1\}$. Then $\Gamma(x^*) = 0$ □

THEOREM 1: $VI(g,\Psi)$ and P_Γ are equivalent.

Proof : $\boxed{\rightarrow}$ Let x^* the optimal solution to $VI(g,\Psi)$. By lemma 1, $\Gamma(x^*)=0$, and since $\Gamma(x) > 0$ for any $x\in\Psi$, then x^* minimizes Γ on Ψ, hence x^* is optimal solution to P_Γ .

$\boxed{\rightarrow}$ Let x^* the optimal solution of P_Γ, and let $\bar{x}$ the one of $VI(g,\Psi)$. By lemma 1, (i) $\Gamma(x^*) > 0$ because $\Gamma(x) > 0$ for any $x\in\Psi$ and $x^*\in\Psi$, and (ii) $\Gamma(\bar{x}) = 0$ because $\bar{x}$ is optimal to $VI(g,\Psi)$. Since $\Gamma(x^*) \leq \Gamma(x)$ for any $x\in\Psi$, in particular $0 \leq \Gamma(x^*) \leq \Gamma(\bar{x}) = 0$, then $\Gamma(x^*) = 0$, and:

$$\Gamma(x^*) = 0 = \max\ g(x^*)^T d \quad \text{s.t. } d\in\Psi_d \cap \{d:\ d^Td=1\}$$

Then for any $d\in\Psi_d \cap \{d:\ d^Td=1\}$, $g(x^*)^Td \leq 0$ (the max). If we suppose that there exists a $\tilde{d}\in\Psi_d \cap \{d:d^Td=1\}$ such that $g(x^*)^T\tilde{d} < 0$, since $-\tilde{d}\in\Psi_d\cap\{d:d^Td=1\}$, $g(x^*)^T(-\tilde{d}) > 0$ which is in contradiction with $0 = \max\ g(x^*)^Td$ $d\in\Psi_d \cap\{d:\ d^Td=1\}$.

Then $g(x^*)^Td = 0$ for any $d\in\Psi_d \cap\{d:\ d^Td=1\}$.

Let $d'\in\Psi_d$:

- If $d'=0$ then $g(x^*)^Td' = 0$
- If $d'\neq 0$ then $d = \frac{d'}{\|d'\|} \in \Psi_d \ \{d:\ d^Td=1\}$

and $g(x^*)^Td = 0 = g(x^*)^Td'\frac{1}{\|d'\|}$, so $g(x^*)^Td' = 0$.

Then, for any $d\in\Psi_d$, $g(x^*)^Td = 0$, and since $d=x-x^*$, then, for any $x\in\Psi$, $g(x^*)^T(x-x^*) = 0$ that proves that x^* is optimal solution to $VI(g,\Psi)$ □

Newton method for $VI(g,\Psi)$ and P_Γ

The usual Newton algorithm used to solve $VI(g,\Psi)$ consists of solving a sequence of variational inequality problems $VI(q^k,\Psi)$, where

$$q^k(x) = g(x^k) + \nabla g(x^k)(x-x^k) \tag{15}$$

is the linear approximation of g at x^k:

Newton: Iteration k: Define x^{k+1} as the solution to $VI(q^k,\Psi)$:

$$\text{find } x^{k+1}\in\Psi \mid q^k(x^{k+1})(x-x^{k+1}) \geq 0 \quad \text{for all } x\in\Psi \tag{16}$$

In order to simplify the notation, we adopt the notation:

$$g_k = g(x^k) \tag{17}$$

$$\text{and} \qquad D_k = \nabla f(x^k) \tag{18}$$

In J.S. Pang (1981), this algorithm is proved, under mild conditions, to have a global quadratic convergence. The next theorem characterizes the optimal solution to each of the subproblems $VI(q^k,\Psi)$:

THEOREM 2: Given $x^k\in\Psi$ and $VI(q^k,\Psi)$ the linear approximation of $VI(g,\Psi)$ evaluated at x^k, the optimal solution x^{k+1} of $VI(q^k,\Psi)$ is

$$x^{k+1} = x^k + d^k \tag{19}$$

where (d^k,λ^k) is the solution of the system:

$$\begin{cases} D_k d^k + A^T\lambda^k = -g_k & (20) \\ Ad^k = 0 & (21) \end{cases}$$

and $q^k(x^{k+1})^T(x-x^{k+1}) = 0$ for all $x\in\Psi$ (i.e. $q^k(x^{k+1}) = (g_k + D_k d^k) \perp \Psi$) .

Proof : • $Ax^{k+1} = A(x^k + d^k) = Ax^k + Ad^k = b+0 = b$, because $x^k \in \Psi$ and $Ad^k = 0$ (equation 21), so $x^{k+1} \in \Psi$.

• $q^k(x^{k+1})^T(x-x^{k+1}) = (g_k + D_k(x^{k+1}-x^k))^T(x-x^{k+1})$ by (15)

$= (g_k + D_k d^k)^T(x-x^{k+1})$ by (19)

$= (g_k + D_k d^k)^T(x-x^{k+1}) + \lambda^{k^T}(b-b)$

$= (g_k + D_k d^k)^T(x-x^{k+1}) + \lambda^{k^T}(Ax-Ax^{k+1})$
for any $x \in \Psi$ and $x^{k+1} \in \Psi$

$= (g_k + D_k d^k)^T(x-x^{k+1}) + \lambda^{k^T}A(x-x^{k+1})$

$= (g_k + D_k d^k + A^T\lambda^k)(x-x^{k+1})$

$= 0^T(x-x^{k+1})$ because of(20)

$= 0 \geq 0$ □

In the following, we show that the Newton direction d^k, defined by (20)-(21), is a descent direction for P_Γ at x^k. In order to do so, we define a gap function Γ_k related to $VI(q^k, \Psi)$ by:

$$\Gamma_k(x) = \begin{cases} \max\limits_d q^k(x)^T d = (g_k + D_k(x-x^k))^T d & (22) \\ \text{s.t. } Ad = 0 & (23) \\ d^T d = 1 & (24) \end{cases}$$

Recall that, by Lemma 1 and theorem 1, P_{Γ_k} : min $\Gamma_k(x)$ s.t. $x \in \Psi$ is equivalent to $VI(q^k, \Psi)$ and that $\Gamma_k(x) \geq 0$ for any $x \in \Psi$, and finally that $\Gamma_k(x^{k+1}) = 0$ where x^{k+1} is the solution to $VI(q^k, \Psi)$.

THEOREM 3: If x^k is not optimal solution of $VI(q^k,\Psi)$, then the Newton direction d^k ((20)-(21)) is a feasible descent direction of $\Gamma_k(x)$ at x^k.

In order to prove theorem 3, we must prove the three next lemmas:

Lemma 2: The projection matrix, on $\Psi_d = \{d: Ad=0\}$, $G = I - A^T(AA^T)^{-1}A$ is idempotent, i.e. $G^2 = G$.

Proof : $GxG = (I - A^T(AA^T)^{-1}A)(I - A^T(AA^T)^{-1}A)$

$$= I - A^T(AA^T)^{-1}A - A^T(AA^T)^{-1} + A^T(AA^T)^{-1}\underbrace{A\cdot A^T(AA^T)^{-1}}_{I}A$$

$$= I - A^T(AA^T)^{-1}A = G$$ □

Lemma 3: The projection matrix G, on Ψ_d, is positive semidefinite but not positive definite.

Proof : For any d: $d^TGd = d^TG^2d$ by lemma 2

$$= d^TG^TGd = \|GD\|_2^2 \geq 0$$

then G is positive semidefinite, but:

for any $d \perp \Psi_d$, the projection Gd of d on Ψ_d is null. Then for any $d\neq 0$, $d \perp \Psi_d$, $d^TGd = 0$ that proves that G is not positive definite □

Lemma 4: If $x^k \varepsilon \Psi$ is not the optimal solution to $VI(q^k,\Psi)$, then $Gg_k \neq 0$.

Proof : Since G is the projection matrix on Ψ_d, $Gg_k = 0$ if and only if $g_k \perp \Psi_d$ that is equivalent to $g_k \perp \Psi$, and that means that $g_k^T(x-x^k) = 0$ for all $x\varepsilon\Psi$, contradicting x^k not optimal solution □

Proof of theorem 3: Let $\Theta \in [0,1[$ and $y(\Theta) = x^k + \Theta d^k = x^k + \bar{d}$.

Consider the solution of

$$\Gamma_k(y(\Theta)) = \max\ (g_k + D_k(x^k + \bar{d} - x^k))^T d$$

$$= \max\ (g_k + D_k \bar{d})^T d$$

$$\text{s.t.}\quad Ad = 0 \quad (\mu: m\times 1)$$

$$d^T d = 1 \quad (\beta = \frac{\alpha}{2}: 1\times 1)$$

The optimal solution $\tilde{d}$ to this problem satisfies the Kuhn-Tucker conditions:

$$-g_k - D_k \bar{d} + A^T \mu + \alpha \tilde{d} = 0 \tag{25}$$

$$A\tilde{d} = 0 \tag{26}$$

$$\tilde{d}^T \tilde{d} = 1 \tag{27}$$

By multiplying (25) by A, we obtain:

$$-A(g_k + D_k \bar{d}) + A^T A\mu + \alpha A\tilde{d} = 0 \tag{28}$$

and since $A\tilde{d} = 0$ (26), (28) becomes:

$$A^T A\ \mu = A(g_k + D_k \bar{d})$$

$$\text{or}\quad \underline{\mu = (A^T A)^{-1} A(g_k + D_k \bar{d})} \tag{29}$$

(Recall that A is of full rank so $A^T A$ is non singular).

By replacing (29) in (25) one obtains:

$$\alpha\tilde{d} = g_k + D_k \bar{d} - A^T(AA^T)^{-1} A\ (g_k + D_k \bar{d})$$

$$\alpha\tilde{d} = (I - A^T(AA^T)^{-1} A)(g_k + D_k \bar{d})$$

$$\underline{\alpha\tilde{d} = G(g_k + D_k \bar{d})} \tag{30}$$

By (27), $\tilde{d}^T\tilde{d} = 1$, so

$\alpha^2 = \alpha^2\tilde{d}^T\tilde{d} = (\alpha\tilde{d})^T(\alpha\tilde{d})$, ans using (30):

$\alpha^2 = (G(g_k + D_k\bar{d}))^T(G(g_k + D_k\bar{d}))$

$\alpha^2 = (g_k + D_k\bar{d})^T G^2 (g_k + D_k\bar{d})$, and since $G^2 = G$ by lemma 2:

$$\alpha^2 = (g_k + D_k\bar{d})^T G (g_k D_k\bar{d}) \tag{31}$$

For the value of the objective $\Gamma_k(y(\Theta))$:

$\alpha\Gamma_k(y(\Theta)) = \alpha(g_k + D_k\bar{d})^T\tilde{d} = (g_k + D_k\bar{d})^T(\alpha\tilde{d})$ and by (30):

$= (g_k + D_k\bar{d})^T G (g_k + D_k\bar{d}) = \alpha^2$ by (31) .

Now, suppose that $\alpha \neq 0$ (to be verified later),

$$\text{then } \Gamma_k(y(\Theta)) = \frac{\alpha^2}{\alpha} = \alpha = [(g_k + D_k\bar{d})^T G (g_k + D_k\bar{d})]^{\frac{1}{2}} \tag{32}$$

and:

$$\begin{aligned}(\Gamma_k(y(\Theta)))^2 &= (g_k + D_k(\Theta d^k))^T G (g_k + D_k(\Theta d^k)) \text{ (because } \bar{d} = \Theta d^k) \\ &= ((1-\Theta)g_k + \Theta(g_k + D_k d^k))^T G ((1-\Theta)g_k + \Theta(g_k + D_k d^k)) \\ &= (1-\Theta)^2 g_k^T G g_k + (1-\Theta)\Theta g_k^T G(g_k + D_k d^k) \\ &\quad + \Theta(1-\Theta)(g_k + D_k d^k)^T G g_k + \Theta^2(g_k + D_k d^k)^T G (g_k + D_k d^k)\end{aligned} \tag{33}$$

By theorem 2, $(g_k + D_k d^k) \perp \Psi$, then its projection $G(g_k + D_k d^k)$ on Ψ_d is null, $G(g_k + D_k d^k) = 0$, so (33) becomes:

$$[\Gamma_k(y(\Theta))]^2 = (1-\Theta)^2 g_k^T G g_k \tag{34}$$

By hypothesis, x^k is not optimal (otherwise the algorithm terminates), so by lemma 4 $Gg_k \neq 0$, then $g_k^T Gg_k = (Gg_k)^T Gg_k = (Gg_k)^2 \neq 0$. It follows that $[\Gamma_k(y(\Theta))]^2 > 0$ for $\Theta \in [0,1[$.

We recall that this was shown under the hypothesis that $\alpha \neq 0$, which is verified since $\alpha^2 = [\Gamma_k(y(\Theta))]^2 > 0$ for $\Theta \in [0,1[$.

Now, for $\Theta=1$, $y(1) = x^k + 1xd^k = x^k + d^k$, and by theorem 2 $(g_k + D_k d^k) \perp \Psi$ so $G(g_k + D_k d^k) = 0$ or $G(g_k + D_k d^k) = (g_k + D_k d^k)^T G (g_k + D_k d^k) = 0$; then $\Gamma_k(y(1)) = 0$.

This is reported on figure 1:

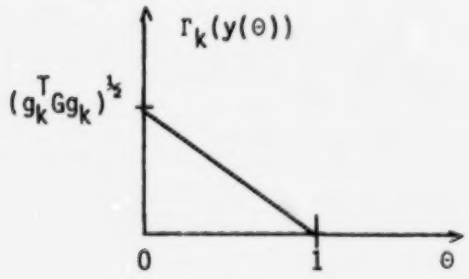

Figure 1: $\Gamma_k(y(\Theta))$ vs $\Theta \in [0,1]$

This proves that Γ_k decreases (strictly) along d^k, for $\Theta \in [0,1]$, and since d^k is feasible, it is a feasible descent direction for Γ_k at x^k .

Next, we are going to use this result to prove the following theorem:

THEOREM 4: If x^k is not optimal for VI(g,Ψ), then the Newton direction d^k is a feasible descent direction for Γ at x^k □

First, we need

Lemma 5: x^k is optimal for VI(g,Ψ), if and only if it is optimal for VI(g^k,Ψ).

Proof : $g(x^k)^T(x-x^k) \geq 0$ for all $x \in \Psi$ <=> $(g(x^k) + D_k(x^k-x^k))^T \geq 0$ for all $x \in \Psi$

<=> $q^k(x^k)^T(x-x^k) \geq 0$ for all $x \in \Psi$ □

Proof of theorem 4: If x^k is not optimal for VI(g,Ψ) (otherwise the algorithm terminates), then, by Lemma 5, x^k is not optimal for VI(q^k,Ψ) .

On the other hand, as in the proof of theorem 3, it can easily be shown that $\Gamma(x) = [g(x)^T G\, g(x)]^{\frac{1}{2}}$

so if we consider x^k and the Newton direction d^k with $\Theta \in]0,1[$:

$$\Gamma^2(x^k + \Theta d^k) = g(x^k + \Theta d^k)^T G\, g(x^k + \Theta d^k) ,$$

and the derivative of Γ^2, relatively to Θ, at x^k(i.e. at Θ=0) is:

$$\nabla = g_k^T G\, D_k d^k + d^{kT} D_k^T g_k .$$

From the proof of theorem 3 (see for instance figure 1), we have that $\Gamma_k^2(y(\Theta)) < g_k^T G\, g_k$ for $\Theta \in]0,1[$

so $(g_k + \Theta D_k d^k)^T G\, (g_k + \Theta D_k d^k) < g_k^T G\, g_k$ for $\Theta \in]0,1[$

so $\underbrace{g_k^T G g_k}_{\bullet} + \underbrace{\Theta g_k^T G D_k d^k + \Theta d^{kT} D_k^T G g_k}_{\Theta\nabla} + \underbrace{\Theta^2 d^{kT} D_k^T G D_k d^k}_{\Theta^2 (G D_k d^k)^2 > 0} < \underbrace{g_k^T G g_k}_{\bullet} \quad \Theta \in]0,1[$

But $\Theta\nabla$ + position number < 0 for $\Theta \in]0,1[$ which means that $\Theta\nabla < 0$ and this proves that d^k is a descent (and feasible) direction for Γ at x^k □

The Newton algorithm, to solve VI(g,Ψ), that consists of:

Newton : Let $x^{k+1} = x^k + d^k$ where (d^k,λ^k) is solution of (20)-(21), may be modified by using theorem 4:

Newton-M: Let $x^{k+1} = x^k + \alpha^k d^k$ where (d^k,λ^k) is solution of (20)-(21) and α^k a step, in the direction d^k, who minimizes the value of Γ at x^k, i.e. $\Gamma(x^k + \alpha^k d^k) = \min_{\alpha} \Gamma(x^k + \alpha d^k)$.

Before terminating this section, we would like to note that in Newton-M, the two formulations VI(g,Ψ) and P_Γ are used; the direction d^k is the Newton direction relative to VI(g,Ψ), and the step α^k reduces the value of the objective function of P_Γ .

CONCLUSION

In this paper, we considered variational inequality of the form VI(g,Ψ): find $x^* \in \Psi \mid g(x^*)^T(x-x^*) \geq 0$ for all $x \in \Psi$ where $\Psi = \{x \in \mathbb{R}^n \mid Ax=b\}$. We proved that this problem is equivalent to a one level mathematical program P_Γ even if the Jacobian ∇_g is not symmetric, where P_Γ: min $\Gamma(x) = g(x)^T G\, g(x)$ s.t. $x \in \Psi$ and G is the projection matrix $I - A^T(AA^T)^{-1}A$.

We considered also the usual Newton method to solve VI(g,Ψ) and proved that the Newton direction is a feasible descent direction for P_Γ; this direction may be used with a step α which minimizes $\Gamma(x^k + \alpha d^k)$ at each iteration.

ACKNOWLEDGEMENTS

This paper benefited from a careful reading by Michael Florian.

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